2.

UNIT – II

I BSC MATHEMATICS

TRIGONOMETRY

Powers of sines and cosines of θ in terms of functions of multiples of θ - expansions of $\sin\theta$ and $\cos\theta$ in a series of ascending powers of θ - Expansions of Inverse circular functions.

chapter III - section 4 and 5.

Expansions for $\cos^n\theta$ and $\sin^n\theta$ in terms of functions of multiples of θ.

We know that
$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$
$$\cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}$$
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

Now we see that $\cos^n\theta$ and $\sin^n\theta$ can be expressed in terms of cosines of multiples of θ or sines of multiples of θ.

Let $x = \cos\theta + i\sin\theta$.

Then $\frac{1}{x} = \cos\theta - i\sin\theta$.

$$\Rightarrow x + \frac{1}{x} = 2\cos\theta \longrightarrow (1)$$

and $$x - \frac{1}{x} = 2i\sin\theta \longrightarrow (2)$$

Also $$x^n = (\cos\theta + i\sin\theta)^n$$
$$= \cos n\theta + i\sin n\theta$$

$$\Rightarrow \frac{1}{x^n} = \cos n\theta - i\sin n\theta.$$

$$\therefore x^n + \frac{1}{x^n} = 2\cos n\theta \longrightarrow (3)$$

and $$x^n - \frac{1}{x^n} = 2i\sin n\theta \longrightarrow (4)$$

Using equations (1), (2), (3) & (4), we expand $\cos^n\theta$ and $\sin^n\theta$ in terms of a series of sine and cosine multiples of θ.

~~Now~~

Expansions of $\cos^n\theta$ when n is a positive integer

Now $2\cos\theta = x + \frac{1}{x}$.

$$\Rightarrow (2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n.$$

$$= x^n + nC_1 x^{n-1}\frac{1}{x} + nC_2 x^{n-2}\frac{1}{x^2} + \cdots + nC_{n-1} x \cdot \frac{1}{x^{n-1}} + \frac{1}{x^n}.$$

$$= \left(x^n + \frac{1}{x^n}\right) + nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \cdots \rightarrow (5).$$

using (3), (5) becomes

$$(2\cos\theta)^n = 2^n \cdot \cos^n\theta = 2\cos n\theta + nC_1 \cdot 2\cos(n-2)\theta + nC_2 \cdot 2\cos(n-4)\theta + \cdots$$

$$\Rightarrow \boxed{\cos^n\theta = \frac{1}{2^{n-1}}\left[\cos n\theta + nC_1\cos(n-2)\theta + nC_2\cos(n-4)\theta + \cdots\right]}$$

Note: (1) If n is odd, there will be even number of terms (ie) $(n+1)$ terms in the expansion of $(x + 1/x)^n$.

These terms can be grouped into $\frac{n+1}{2}$ pairs.

The last term contains $\cos\theta$.

$\Rightarrow$ Coefficient of $\cos\theta$ in the expansion of

$$2^{n-1}\cos^n\theta = nC_{\frac{(n-1)}{2}}.$$

Note: (2) If n is even, there will be odd number of terms ~~(ie)~~ in the expansion of $(x + 1/x)^n$.

These terms can be grouped in $\frac{n}{2}$ pairs, leaving the middle term which is independent of x.

Hence the last term in the expansion of $2^{n-1}\cos^n\theta$ which is independent of $\theta = \frac{nC_{\frac{n}{2}}}{2}$.

Expansion of $\sin^n\theta$ when n is a positive integer

Now $2i\sin\theta = x - 1/x$.

$\Rightarrow (2i\sin\theta)^n = (x - 1/x)^n$

$= x^n - nC_1 x^{n-1}\cdot\frac{1}{x} + nC_2\cdot x^{n-2}\cdot\frac{1}{x^2} - nC_3 x^{n-3}\cdot\frac{1}{x^3} + \cdots$

Case (i) When n is even.

The number of terms in the expansions is odd, The signs of the terms are alternatively positive and negative and the last term is positive.

$\therefore (2i\sin\theta)^n = \left(x^n + \frac{1}{x^n}\right) - nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \cdots$

i.e $2^n (i)^n \sin^n\theta = 2\cos n\theta - nC_1\, 2\cos(n-2)\theta + nC_2 \cos(n-4)\theta - \cdots$

$\Rightarrow (-1)^{\frac{n}{2}} 2^{n-1}\sin^n\theta = \cos n\theta - nC_1 \cos(n-2)\theta + nC_2 \cos(n-4)\theta - \cdots$

Case (ii) When n is odd

$(2i\sin\theta)^n = x^n - nC_1 x^{n-2} + nC_2 x^{n-4} - \cdots - \frac{1}{x^n}$.

$= \left(x^n - \frac{1}{x^n}\right) - nC_1\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + nC_2\left(x^{n-4} - \frac{1}{x^{n-4}}\right) \cdots$

$= 2i\sin n\theta - nC_1\, 2i\sin(n-2)\theta + nC_2\, 2i\sin(n-4)\theta - \cdots$

i.e $2^{n-1}(i)^{n-1}\sin^n\theta = \sin n\theta - nC_1 \sin(n-2)\theta + nC_2 \sin(n-4)\theta - \cdots$

$\Rightarrow 2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - nC_1\sin(n-2)\theta + nC_2\sin(n-4)\theta - \cdots$

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Problems.

1 (a) Expand (i) $\cos^6\theta$ (ii) $\cos^5\theta$ in a series of cosines of multiples of θ.

Solution:

Let $x = \cos\theta + i\sin\theta$.

$x + 1/x = 2\cos\theta$.

(i) $\cos^6\theta$

Now $(2\cos\theta)^6 = (x + 1/x)^6$.

$= x^6 + 6C_1 x^5 \cdot 1/x + 6C_2 x^4 \cdot \frac{1}{x^2} + 6C_3 x^3 \cdot \frac{1}{x^3} + 6C_4 x^2 \cdot \frac{1}{x^4} + 6C_5 x \cdot \frac{1}{x^5} + 6C_6 \cdot \frac{1}{x^6}$.

$= \left(x^6 + \frac{1}{x^6}\right) + 6C_1\left(x^4 + \frac{1}{x^4}\right) + 6C_2\left(x^2 + \frac{1}{x^2}\right) + 6C_3$.

$= 2\cos 6\theta + 6C_1 \cdot 2\cos 4\theta + 6C_2 \cdot 2\cos 2\theta + 6C_3$.

$\Rightarrow 2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

$\Rightarrow \boxed{\cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).}$

Also, $(2\cos\theta)^5 = (x + 1/x)^5$.

$= x^5 + 5C_1 x^4 \cdot \frac{1}{x} + 5C_2 \cdot x^3 \cdot 1/x^2 + 5C_3 x^2 \cdot 1/x^3 + 5C_4 \cdot x \cdot \frac{1}{x^4} + 5C_5 \cdot \frac{1}{x^5}$.

$= \left(x^5 + \frac{1}{x^5}\right) + 5C_1\left(x^3 + 1/x^3\right) + 5C_2\left(x + 1/x\right)$.

$= 2\cos 5\theta + 5C_1(2\cos 3\theta) + 5C_2(2\cos\theta)$.

$\Rightarrow 2^4\cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta$.

(1) (b) Expand $\sin^7\theta$ in a series of sines of multiples of θ.

Solution:

We have

$$\left(x-\frac{1}{x}\right)^7 = x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}.$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - 1/x^3\right) - 35\left(x - 1/x\right).$$

Since $x - 1/x = 2i\sin\theta$, & $x^n - 1/x^n = 2i\sin n\theta$, we have

$$(2i\sin\theta)^7 = 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta).$$

$$\Rightarrow 2^6(-1)^3\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$$

$$\Rightarrow \sin^7\theta = -\frac{1}{64}\left(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta\right).$$

(1) (c) Expand $\sin^6\theta$ in a series of cosines of multiples of θ.

Solution:

$$(x - 1/x)^6 = x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + 1/x^6.$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 6\left(x^4 + 1/x^4\right) + 15\left(x^2 + 1/x^2\right) - 20. \quad \cancel{\pm 1/x}$$

Since $x + 1/x = 2\cos\theta$, $x^n + 1/x^n = 2\cos n\theta$, $x - 1/x = 2i\sin\theta$,

we have

$$(2i\sin\theta)^6 = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20.$$

$$\Rightarrow 2^5(-1)^3\sin^6\theta = \cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10.$$

$$\Rightarrow \sin^6\theta = -\frac{1}{32}\left(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10\right).$$

(2) Prove the following results:

(a) $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$

Solution: $x = \cos\theta + i\sin\theta$

$$x + 1/x = 2\cos\theta,$$

$$x^n + 1/x^n = 2\cos n\theta.$$

Now $(x+1/x)^6 = x^6 + 6C_1 x^4 + 6C_2 x^2 + 6C_3 + 6C_4 \frac{1}{x^2} + 6C_5 1/x^4 + 6C_6 \frac{1}{x^6}.$

$$= (x^6 + 1/x^6) + 6C_1(x^4 + 1/x^4) + 6C_2(x^2 + 1/x^2) + 6C_3.$$

$$\Rightarrow (2\cos\theta)^6 = 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20.$$

$$\Rightarrow 32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$$

Note: The coefficients of $(x+1/x)^6$ expansion are given in the following scheme called Pascal's triangle.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

$$\therefore (x+1/x)^6 = (x^6 + 1/x^6) + 6(x^4 + 1/x^4) + 15(x^2 + 1/x^2) + 20..$$

2 (b) $-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$

Solution:

$x = \cos\theta + i\sin\theta, \quad x + 1/x = 2\cos\theta, \quad x - 1/x = 2i\sin\theta.$

$x^n + 1/x^n = 2\cos n\theta, \quad x^n - 1/x^n = 2i\sin n\theta.$

$$\left(x - \frac{1}{x}\right)^7 = x^7 - 7C_1x^5 + 7C_2x^3 - 7C_3x + 7C_4\frac{1}{x} - 7C_5\frac{1}{x^3} + 7C_6\frac{1}{x^5} - \frac{1}{x^7}.$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - 1/x^3\right) - 35\left(x - 1/x\right).$$

$$\Rightarrow (2i\sin\theta)^7 = 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta).$$

$$\Rightarrow -2^6\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$$

$$\Rightarrow -64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$$

2 (c) Show that $128\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35.$

Solution:

$x = \cos\theta + i\sin\theta, \quad x + 1/x = 2\cos\theta, \quad x - 1/x = 2i\sin\theta.$

$x^n + \frac{1}{x^n} = 2\cos n\theta, \quad x^n - \frac{1}{x^n} = 2i\sin n\theta.$

Consider $(x - 1/x)^8$.

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

The Coefficients is $\left(x-\frac{1}{x}\right)^8$ are alternatively positive and negative.

$$\therefore \left(x-\frac{1}{x}\right)^8 = x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - 56\cdot\frac{1}{x^2} + 28\cdot\frac{1}{x^4} - 8\cdot\frac{1}{x^6} + \frac{1}{x^8}$$

$$= \left(x^8+\frac{1}{x^8}\right) - 8\left(x^6+\cancel{x^6}\frac{1}{x^6}\right) + 28\left(x^4+\frac{1}{x^4}\right) - 56\left(x^2+\frac{1}{x^2}\right) + 70.$$

$$\Rightarrow (2i\sin\theta)^8 = 2\cos 8\theta - 8\cdot(2\cos 6\theta) + 28(2\cos 4\theta) - 56(2\cos 2\theta) + 70.$$

$$\Rightarrow 128\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35.$$

(2) (d) $2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta.$

(e) $2^7\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35.$

(f) establish the formula $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta.$

(g) $2\cos^9\theta = \frac{1}{128}\left[\cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta,\right.$

(h) $2^8\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta.$

(i) $\cos^4\theta = \cos 4\theta - 4\cos 2\theta + 6.$

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(3) (a) Expand $\sin^3\theta\cos^5\theta$ in a series of sines of multiples of θ.

Solution:

$x = \cos\theta + i\sin\theta, \quad x + 1/x = 2\cos\theta, \quad x - 1/x = 2i\sin\theta.$

$x^n + 1/x^n = 2\cos n\theta, \quad x^n - 1/x^n = 2i\sin n\theta.$

Now $(2i\sin\theta)^3(2\cos\theta)^5 = (x - 1/x)^3(x + 1/x)^5$

$$= \left(x - \frac{1}{x}\right)^3 (x + 1/x)^3 (x + 1/x)^2$$

$$= (x^2 - 1/x^2)^3 (x + 1/x)^2$$

$$= \left(x^6 - 3x^4 \frac{1}{x^2} + 3x^2 \cdot \frac{1}{x^4} - \frac{1}{x^6}\right)\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= x^8 + 2x^6 + x^4 - 3x^4 - 6x^2 - 3 + 3 + \frac{6}{x^2} + \frac{3}{x^4} - \frac{1}{x^4} - \frac{2}{x^6} - \frac{1}{x^8}$$

$$= \left(x^8 - \frac{1}{x^8}\right) + 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) - 6(x^2 - 1/x^2)$$

$\Rightarrow 2^8 i^3 \sin^3\theta\cos^5\theta = \cancel{2\cos 8\theta + 2(2\cos 6\theta) - 2(2\cos 4\theta) - 6(2\cos 2\theta)}$

$2i\sin 8\theta + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta)$

$= 2i\sin 8\theta + 2(2i\sin 6\theta) - 2(2i\sin 4\theta) - 6(2i\sin 2\theta)$

$= 2i\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta.$

$\Rightarrow -2^7\sin^3\theta\cos^5\theta = \sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta$

$\Rightarrow \sin^3\theta\cos^5\theta = -\frac{1}{2^7}[\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta].$

(3) (b) Expand $\sin^4\theta\cos^2\theta$ in a series of cosines of multiples of θ.

Solution: $x = \cos\theta + i\sin\theta, \quad x + 1/x = 2\cos\theta.$

$x - 1/x = 2i\sin\theta, \quad x^n + 1/x^n = 2\cos n\theta, \quad x^n - 1/x^n = 2i\sin n\theta.$

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$$(2i\sin\theta)^4(2\cos\theta)^2 = (x-1/x)^4(x+1/x)^2$$

$$= (x^2-1/x^2)^2(x-1/x)^2$$

$$= \left(x^4-2+\frac{1}{x^4}\right)(x^2-2+1/x^2)$$

$$= x^6-2x^4+x^2-2x^2+4-\frac{2}{x^2}+\frac{1}{x^2}-\frac{2}{x^4}+\frac{1}{x^6}.$$

$$= \left(x^6+\frac{1}{x^6}\right)-2\left(x^4+\frac{1}{x^4}\right)-2(x^2+1/x^2)+4.$$

$$= 2\cos 6\theta - 2\cdot(2\cos 4\theta) - 2(2\cos 2\theta)+4.$$

$$\Rightarrow 2^6 i^4 \sin^4\theta\cos^2\theta = 2\cos 6\theta - 2\cos 4\theta - 2\cos 2\theta + 2].$$

$$\Rightarrow \sin^4\theta\cos^2\theta = \frac{1}{2^6}[\cos 6\theta - 2\cos 4\theta$$

Note: In the expansion of the product $(x-1/x)^4(x+1/x)^2$, first write the pascal triangle for the coefficients $(x-1/x)^4$. Signs are +ve & −ve alternatively in the 4th row. Then write 2 more rows of pascal triangle. The last row gives the coefficients of the expression $(x-1/x)^4(x+1/x)^2$.

1 1

1 2 1

1 3 3 1

1 −4 6 −4 +1

1 −3 2 2 −3 1

1 −2 −1 4 −1 −2 1

$$\therefore (x-1/x)^4(x+1/x)^2 = x^6-2x^4-x^2+4-\frac{1}{x^2}-\frac{2}{x^4}+\frac{1}{x^6}.$$

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3 (b) Expand $\sin^3\theta\cos^4\theta$ in terms of sines of multiples of θ.

Solution:

Consider $(2i\sin\theta)^3(2\cos\theta)^4 = (x-1/x)^3(x+1/x)^4$

$$= (x-1/x)^3(x+1/x)^3(x+1/x)$$

$$= (x^2-1/x^2)^3(x+1/x)$$

$$= \left(x^6-3x^2+\frac{3}{x^2}-\frac{1}{x^6}\right)(x+1/x)$$

$$= x^7+x^5-3x^3-3x+\frac{3}{x}+\frac{3}{x^3}-\frac{1}{x^5}-\frac{1}{x^7}$$

$$= (x^7-1/x^7)+(x^5-1/x^5)-3\left(x^3-\frac{1}{x^3}\right)-3(x-1/x)$$

$$= 2i\sin7\theta+2i\sin5\theta-3\cdot2i\sin3\theta-3\cdot2i\sin\theta$$

$$2^7 i^3\sin^3\theta\cos^4\theta = 2i\sin7\theta+2i\sin5\theta-3\cdot2i\sin3\theta-3\cdot2i\sin\theta$$

$$\sin^3\theta\cos^4\theta = -\frac{1}{2^6}\left[\sin7\theta+\sin5\theta-3\sin3\theta-3\sin\theta\right]$$

Note:

Pascal's triangle for $(x-1/x)^3(x+1/x)^4$

1 1

1 2 1

1 −3 3 −1

1 −2 0 2 −1

1 −1 −2 2 1 −1

1 0 −3 0 3 0 −1

1 1 −3 −3 3 3 −1 −1

(3)(d) Prove $2^5 \sin^4\theta \cos^2\theta = \cos 6\theta - 2\cos 4\theta + 2$.

(3)(e) Prove $64 \sin^5\theta \cos^2\theta = \sin 7\theta - 3\sin 5\theta + \sin 3\theta + 5\sin\theta$.

(f) $\sin^6\theta \cos^2\theta = -\frac{1}{2^7}\left[\cos 8\theta - 4\cos 6\theta + 4\cos 4\theta + 4\cos 2\theta - 5\right]$

(g) $\sin^7\theta \cos^2\theta = -\frac{1}{2^8}\left[\sin 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta\right]$.

(h) Express $\sin^5\theta \cos^2\theta$ as the sum of sine of multiples of θ.

(i) Expand (*) $\cos^7\theta \sin^3\theta$ in a series of multiples of θ.
(**) $\sin\theta \cos^5\theta$ (***) $\sin^7\theta \cos^3\theta$

(j) Expand (*) $\cos^4\theta \sin^4\theta$ as a series of cosines of multiples of θ.
(**) $\cos^6\theta \sin^6\theta$
(***) $\sin^2\theta \cos\theta$

④ Prove that

ⓐ $64(\cos^8\theta+\sin^8\theta)=\cos 8\theta+28\cos 4\theta+35.$

Solution:

$$\left(x+\frac{1}{x}\right)^8=x^8+8C_1x^6+8C_2x^4+8C_3x^2+8C_4+8C_5\cdot{}^1\!/_{x^2}+8C_6\frac{1}{x^4}+8C_7\frac{1}{x^6}+\frac{1}{x^8}.$$

$$\Rightarrow (2\cos\theta)^8=\left(x^8+{}^1\!/_{x^8}\right)+8\left(x^6+{}^1\!/_{x^6}\right)+28\left(x^4+{}^1\!/_{x^4}\right)+56\left(x^2+{}^1\!/_{x^2}\right)+70.$$

$$\Rightarrow 2^8\cos^8\theta=2\cos 8\theta+8\cdot 2\cos 6\theta+28\cdot 2\cos 4\theta+56(2\cos 2\theta)+70.$$

$$\Rightarrow 128\cos^8\theta=\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35.$$

Similarly, Considering $(x-{}^1\!/_x)^8$, we get

$$128\sin^8\theta=\cos 8\theta-8\cos 6\theta+28\cos 4\theta-56\cos 2\theta+35.$$

Adding both, we get

$$128(\cos^8\theta+\sin^8\theta)=2(\cos 8\theta+28\cos 4\theta+35).$$

$$\Rightarrow 64(\cos^8\theta+\sin^8\theta)=\cos 8\theta+28\cos 4\theta+35.$$

4ⓑ Show that $\sin^6\theta+\cos^6\theta=\frac{1}{8}(5+3\cos 4\theta).$

Expansions of $\sin\theta$ and $\cos\theta$ in a series of ascending powers of θ.

We have shown that when n is a positive integer,

$$\cos n\alpha = \cos^n\alpha - nC_2 \cos^{n-2}\alpha \sin^2\alpha + nC_4 \cos^{n-4}\alpha \sin^4\alpha - \cdots \rightarrow (1)$$

Let us take $n\alpha = \theta \Rightarrow \boxed{\alpha = \theta/n}$

$$\therefore (1) \Rightarrow \cos\theta = \cos^n\left(\frac{\theta}{n}\right) - \frac{n(n-1)}{2!}\cos^{n-2}\left(\frac{\theta}{n}\right)\sin^2(\theta/n) + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}(\theta/n)\cdot\sin^4(\theta/n) - \cdots$$

$$= \cos^n(\theta/n) - \frac{n(n-1)}{2!}\cos^{n-2}(\theta/n)\left(\frac{\sin\theta/n}{\theta/n}\right)^2(\theta/n)^2 + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}(\theta/n)\left(\frac{\sin\theta/n}{\theta/n}\right)^4(\theta/n)^4 - \cdots$$

$$= \cos^n(\theta/n) - \frac{1.(1-1/n)}{2!}\theta^2\cos^{n-2}(\theta/n)\left(\frac{\sin\theta/n}{\theta/n}\right)^2 + \frac{1.(1-1/n)(1-2/n)(1-3/n)}{4!}\theta^4\cos^{n-4}(\theta/n)\left(\frac{\sin\theta/n}{\theta/n}\right)^4 - \cdots$$

As $n \to \infty$, $\theta/n \to 0$ and $\frac{\sin\theta/n}{\theta/n} \to 1$.

$$\therefore \boxed{\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \infty}$$

When n is a positive integer,

$$\sin n\alpha = nC_1 \cos^{n-1}\alpha \sin\alpha - nC_3 \cos^{n-3}\alpha \sin^3\alpha + nC_5 \cos^{n-5}\alpha \sin^5\alpha - \cdots$$

Put $n\alpha = \theta$.

$$\Rightarrow \sin n\theta = nC_1 \cos^{n-1}(\theta/n)\sin(\theta/n) - nC_3 \cos^{n-3}(\theta/n)\sin^3(\theta/n) + \cdots$$

$$= nC_1 \cos^{n-1}(\theta/n)\left(\frac{\sin\theta/n}{\theta/n}\right)\theta/n - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}(\theta/n)\left(\frac{\sin\theta/n}{\theta/n}\right)^3(\theta/n)^3 + \cdots$$

$$= \cos^{n-1}(\theta/n)\left(\frac{\sin\theta/n}{\theta/n}\right)\theta - 1\frac{(1-1/n)(1-2/n)}{3!}\cos^{n-3}\theta/n\left(\frac{\sin\theta/n}{\theta/n}\right)^3\theta^3 + \cdots$$

As $n \to \infty$, $\theta/n \to 0$ and $\frac{\sin\theta/n}{\theta/n} \to 1$.

$$\therefore \boxed{\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \infty}$$

Note: The expressions are valid only if θ is expressed in radians.
The expansion for $\sin\theta$ and $\cos\theta$ are infinite.

Expansion of $\tan\theta$.

Assuming the expansion for $\sin\theta$ and $\cos\theta$, we get the series for $\tan\theta$, upto the term containing θ^5.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

$$= \frac{\theta - \theta^3/3! + \theta^5/5! - \cdots \infty}{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots \infty}.$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \cdots\right)\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right)^{-1}.$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}\right)\left[1 - \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right)\right]^{-1}$$ neglecting terms beyond θ^5.

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}\right)\left[1 + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right) + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!}\right)^2 + \cdots\right].$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}\right)\left[1 + \frac{\theta^2}{2} - \frac{\theta^4}{24} + \frac{\theta^4}{4} + \frac{\theta^8}{(24)^2} - 2\cdot\frac{\theta^6}{48} \cdots\right]$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}\right)\left(1 + \frac{\theta^2}{2} - \frac{\theta^4}{24} + \frac{\theta^4}{4}\right).$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}\right)\left(1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24}\right).$$

$$= \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \frac{\theta^3}{2} - \frac{\theta^5}{12} + \frac{5\theta^5}{24} \cancel{+ \frac{5\theta^5}{24}}.$$

$$= \theta + \frac{2\theta^3}{6} + \theta^5\left(\frac{1}{120} - \frac{1}{12} + \frac{5}{24}\right).$$

$$= \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}.$$

$$\therefore \boxed{\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15}}$$

$$\theta^3\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$\theta^3\left(\frac{3-1}{6}\right) = \frac{2\theta^3}{6}$$

$$\theta^5\left(\frac{1}{120} + \frac{5}{24} - \frac{1}{12}\right)$$

$$= \theta^5\,\frac{1+25-10}{120}$$

$$= \frac{16\theta^5}{120} = \frac{2\theta^5}{15}$$

Problems.

(5) (a) Find $\lim_{\theta \to 0} \dfrac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)}$

Solution:

$$\frac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)} =$$

$$= \frac{n\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right) - \left(n\theta - \frac{n^3\theta^3}{3!} + \frac{n^5\theta^5}{5!} \cdots\right)}{\theta\left[\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) - \left(n\theta - \frac{n^3\theta^3}{3!} + \frac{n^5\theta^5}{5!} \cdots\right)\right]}$$

$\sin\theta = \theta - \dfrac{\theta^3}{3!} + \dfrac{\theta^5}{5!} - \cdots$

$\cos\theta = 1 - \dfrac{\theta^2}{2!} + \dfrac{\theta^4}{4!} - \cdots$

$\tan\theta = \theta + \dfrac{\theta^3}{3} + \dfrac{2\theta^5}{15}.$

$$= \frac{\cancel{n\theta} - \frac{n\theta^3}{3!} + \frac{n\theta^5}{5!} \cdots - \cancel{n\theta} + \frac{n^3\theta^3}{3!} - \frac{n^5\theta^5}{5!} + \cdots}{\theta\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - n\theta + \frac{n^3\theta^3}{3!} - \frac{n^5\theta^5}{5!} + \cdots\right]}$$

$$= \frac{\frac{\theta^3}{3!}(n^3 - n) + \frac{\theta^5}{5!}(n - n^5) - \cdots}{1 - n\theta - \frac{\theta^2}{2!} + \frac{n^3\theta^3}{3!} + \frac{\theta^4}{4!} - \frac{n^5\theta^5}{5!} + \cdots}$$

Now $\lim_{\theta \to 0} \dfrac{n\sin\theta - \sin n\theta}{\theta(\cos\theta - \sin n\theta)} = \dfrac{0}{1} = 0.$

(5) (b) Find $\lim_{\theta \to 0} \dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}.$

Solution:

$$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta} - \frac{1}{\cos\theta} + 1}$$

$$= \frac{\left(\frac{\sin\theta + 1 - \cos\theta}{\cos\theta}\right)}{\left(\frac{\sin\theta - 1 + \cos\theta}{\cos\theta}\right)}$$

$$= \frac{\sin\theta + 1 - \cos\theta}{\sin\theta - 1 + \cos\theta}$$

$$= \frac{\left(\theta - \frac{\theta^3}{3!} + \cdots\right) + 1 - \left(1 - \theta^2/2! + \cdots\right)}{\left(\theta - \frac{\theta^3}{3!} + \cdots\right) - 1 + \left(1 - \frac{\theta^2}{2!} + \cdots\right)}.$$

$$= \frac{\theta + \frac{\theta^2}{2!} - \frac{\theta^3}{3!} \cdots}{\theta - \frac{\theta^2}{2!} - \theta^3/3! \cdots}$$

$$= \frac{1 + \frac{\theta}{2!} - \frac{\theta^2}{3!} \cdots}{1 - \frac{\theta}{2!} - \frac{\theta^2}{3!} \cdots}$$

$$\therefore \lim_{\theta \to 0} \frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1}{1} = 1.$$

5 c Evaluate $\lim_{x \to 0} \frac{\tan 2x - 2\tan x}{x^3}$.

Solution:

$$\lim_{x \to 0} \frac{\tan 2x - 2\tan x}{x^3} = \lim_{x \to 0} \frac{2x + \frac{(2x)^3}{3} + \frac{2}{15}(2x)^5 + \cdots - 2\left[x + x^3/3 + \frac{2x^5}{15} + \cdots\right]}{x^3}$$

$$= \lim_{x \to 0} \frac{\cancel{2x} + \frac{8x^3}{3} + \frac{64}{15}x^5 + \cdots - \cancel{2x} - \frac{2x^3}{3} - \frac{4x^5}{15} \cdots}{x^3}$$

$$= \lim_{x \to 0} \frac{\left(\frac{8x^3}{3} - \frac{2x^3}{3}\right) + \left(\frac{64}{15}x^5 - \frac{4}{15}x^5\right) + \cdots \infty}{x^3}$$

$$= \lim_{x \to 0} \frac{\frac{6x^3}{3} + \frac{60}{15}x^5 + \cdots \infty}{x^3} = \lim_{x \to 0} 2 + 4x^2 + \cdots \infty$$

$$= 2 \,//$$

5d Evaluate $\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$.

Solution:

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x}$$

$$= \lim_{x \to 0} \frac{\left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \infty\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty\right)}{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty\right)^3}.$$

$$= \lim_{x \to 0} \frac{\cancel{x} + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \infty - \cancel{x} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots \infty}{\cancel{x}^3\left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \infty\right)^3}$$

$$= \lim_{x \to 0} \frac{x^3\left(\frac{1}{3} + \frac{1}{3!}\right) + x^5\left(\frac{2}{15} - \frac{1}{5!}\right) + \cdots}{x^3\left(1 - \frac{x^2}{2!} + \cdots\right)^3}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} + x^2\left(\frac{2}{15} - \frac{1}{5!}\right) + \cdots}{\left(1 - \frac{x^2}{2!} + \cdots\right)^3}$$

$$= \frac{1}{2}.$$

Evaluate

5e $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$

5f $\lim_{x \to 0} \frac{\sin 2x - 2\sin x}{x^3}$.

5g $\lim_{x \to 0} \frac{\sin x - x\cos x}{x^3}$

5h $\lim_{x \to 0} \frac{\tan 2x - x\cos x}{x^3}$

5i $\lim_{x \to 0} \frac{\sin x - \tan x}{x^2}$

6a) Determine a, b, c such that

$$\lim_{\theta \to 0} \frac{\theta(a+b\cos\theta) - c\sin\theta}{\theta^5} = 1.$$

Solution:

$$\lim_{\theta \to 0} \frac{\theta(a+b\cos\theta) - c\sin\theta}{\theta^5} = 1.$$

$$\Rightarrow \lim_{\theta \to 0} \frac{\theta\left[a+b\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots\right)-c\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\cdots\right)\right]}{\theta^5} = 1$$

$$\Rightarrow \lim_{\theta \to 0} \frac{\left[\theta(a+b-c)+\theta^3\left(-\frac{b}{2!}+\frac{c}{3!}\right)+\theta^5\left(\frac{b}{4!}-\frac{c}{5!}\right)+\cdots\right]}{\theta^5} = 1$$

Equating the Coefficients of θ, θ^3 & θ^5 on both sides we get

$$a+b\cancel{+}c = 0 \rightarrow (1)$$

$$-\frac{b}{2}+\frac{c}{6} = 0 \rightarrow (2)$$

$$\frac{b}{24}-\frac{c}{120} = 1 \rightarrow (3)$$

$(2) \Rightarrow 3b - c = 0 \rightarrow (4)$

$(3) \Rightarrow 5b - c = 120 \rightarrow (5)$

$(5) - (4) \Rightarrow 2b = 120 \Rightarrow \boxed{b=60}$

$\therefore c = 3b = 180 \Rightarrow \boxed{c=180}$

$a+b\cancel{+}c = 0.$

$(1) \Rightarrow a = -b+c$

$= -60+180$

$= 120$

$\Rightarrow \boxed{a=120}$

$\therefore a=120, \; b=60, \; c=180.$

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(6b) Determine a and b so that as $\theta \to 0$

$$\lt_{\theta \to 0} \frac{a - \theta \sin\theta - b\cos\theta}{\theta^4} = \frac{1}{12}.$$

(7)(a) Evaluate $\lim_{x \to \pi/2} \frac{\sin x + \cos 2x}{\cos^2 x}$.

Solution:

Put $x = \theta + \pi/2$.

As $x \to \pi/2$, $\theta \to 0$.

$$\lim_{x \to \pi/2} \frac{\sin x + \cos 2x}{\cos^2 x} = \lt_{\theta \to 0} \frac{\sin(\theta + \pi/2) + \cos(2\theta + \pi)}{\cos^2(\theta + \pi/2)}.$$

$$= \lt_{\theta \to 0} \frac{\cos\theta - \cos 2\theta}{\sin^2\theta}.$$

$$= \lt_{\theta \to 0} \frac{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) - \left(1 - \frac{4\theta^2}{2!} + \frac{16\theta^4}{4!} + \cdots\right)}{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right)^2}$$

$$= \lt_{\theta \to 0} \frac{\frac{3}{2}\theta^2 - \frac{5}{8}\theta^4 - \cdots}{\theta^2\left(1 - \theta^2/3! + \frac{\theta^4}{5!} \cdots\right)^2}$$

$$= \lt_{\theta \to 0} \frac{\frac{3}{2} - \frac{5}{8}\theta^2 - \cdots}{\left(1 - \theta^2/3! + \frac{\theta^4}{5!} \cdots\right)^2}$$

$$= 3/2. \quad //$$

7(b) Evaluate $\lt_{x \to \pi/2} (\sec x - \tan x)$.

7(c) Show that $\lt_{\theta \to \pi/2} \frac{\cos\theta - \sin 2\theta}{\sin 3\theta} = 1/3$.

(8) (a) Solve approximately in radians.

$\sin(\pi/3 + x) = 0.87$

Solution:

$\sin(\pi/3 + x) = 0.87$

$\Rightarrow \sin\pi/3 \cos x + \cos\pi/3 \sin x = 0.87$

$\Rightarrow \sin\pi/3 = \frac{\sqrt{3}}{2} \simeq 0.87,$ $\left[\frac{1.732}{2}\right]$

x is small.

Taking approximations, we get

$\frac{\sqrt{3}}{2} + \frac{1}{2} \cdot x = 0.87$

$\sqrt{3} + x = 1.74$

$\Rightarrow x = 1.74 - \sqrt{3}$

$\Rightarrow = 1.74 - 1.732$

$\Rightarrow x = 0.008$ radian approximately

(8) (b) If $\cos(\pi/3 + \theta) = 0.49$ show that $\theta = 39' 5''$ nearly.

Solution:

$\cos(\pi/3 + \theta) = 0.49$

$\Rightarrow \cos\pi/3 \cos\theta - \sin\pi/3 \sin\theta = 0.49$

$\Rightarrow \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = 0.49$

$\Rightarrow \frac{1}{2}\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) - \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \cdots\right) = 0.49.$

$\Rightarrow \frac{1}{2} - \frac{\sqrt{3}}{2}\theta = 0.49$ nearly.

$\Rightarrow 1 - \sqrt{3}\theta = 0.98$

$1 - 0.98 = \sqrt{3}\theta$

$\Rightarrow 0.02 = \sqrt{3}\theta \Rightarrow \theta = \frac{0.02}{\sqrt{3}}$

$\Rightarrow \theta = 0.01155$ radians.

$\Rightarrow \boxed{\theta = 39' 5'' \text{ nearly}}$

8(c) If $\sin(\pi/6 + \theta) = 0.51$, find the approximate value of θ.

(9) (a) If $\frac{\sin x}{x} = \frac{863}{864}$, find an approximate value of x.

Solution:

$$\frac{\sin x}{x} = \frac{863}{864}.$$

$$\Rightarrow \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} = \frac{863}{864}.$$

$$\Rightarrow 1 - \frac{x^2}{6} + \cdots = \frac{863}{864}.$$

$$\Rightarrow \frac{x^2}{6} = 1 - \frac{863}{864} = \frac{1}{864}, \text{ approximately.}$$

$$\Rightarrow x^2 = \frac{6}{864} \text{ approximately.}$$

$$\Rightarrow x^2 = \frac{1}{144} \Rightarrow x = \frac{1}{12}$$

$$x = \frac{1}{12} \times 57^\circ 17' 44''$$

$$= 4^\circ 46' 29''.$$

(9) (b) If $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, show that $\theta = 1^\circ 58'$ approximately.

Solution: $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$, θ is measured in radians.

As $\frac{5045}{5046}$ is approximately equal to 1, θ is very small.

$$\therefore \frac{\sin\theta}{\theta} = \frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots}{\theta} = 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} \text{ approximately.}$$

$$\therefore 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} = \frac{5045}{5046}.$$

$$\Rightarrow \frac{\theta^2}{6} - \frac{\theta^4}{120} = 1 - \frac{5045}{5046} = \frac{1}{5046}.$$

$$\Rightarrow \theta^2 = \frac{6}{5046} = \frac{1}{841}.$$

$$\Rightarrow \theta = \frac{1}{29} \text{ of a radian.}$$

$$= \frac{1}{29} \text{ of } 57^\circ 17' 44.8''$$

$$= 1^\circ 58' \text{ approximately.}$$

9 ⓒ If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$, find θ approximately.

Solution:

$$\frac{\tan\theta}{\theta} = \frac{2524}{2523}$$

$$\Rightarrow \frac{\theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots}{\theta} = \frac{2524}{2523}.$$

$$\Rightarrow 1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \cdots = 1 + \frac{1}{2523}$$

approximately.

$$\Rightarrow \frac{\theta^2}{3} = \frac{1}{2523}$$

$$\Rightarrow \theta^2 = \frac{3}{2523} = \frac{1}{841}$$

$$\Rightarrow \theta = \frac{1}{29} \text{ radians}$$

$$\Rightarrow \theta = 1^\circ 58' \text{ nearly.}$$

9d Given that $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, show that θ is nearly the circular measure of 3°.

9e If $\frac{\sin\theta}{\theta} = \frac{19493}{19494}$ ~~$\frac{5045}{5046}$~~, show that $\theta = 1^\circ$ approximately

9f If $\frac{\sin\theta}{\theta} = \frac{5765}{5766}$, show that $\theta = 1^\circ 51'$ approximately.

9g Find θ approximately to the nearest minute, if $\cos\theta = \frac{1681}{1682}$ taking $\pi = 3.14159$.

(10) If $\sin\theta = 0.5033$, show that θ is approximately $30^\circ\ 13'\ 6''$

Solution:

$$\sin\theta = 0.5033$$

Since $\sin\theta$ is nearly $1/2$,

$\theta = \pi/6 + x$, where x is small.

$\therefore \sin(\pi/6 + x) = 0.5033$

$\Rightarrow \sin\pi/6 \cos x + \cos\pi/6 \sin x = 0.5033$

$\Rightarrow \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x = 0.5033$

As x is small, $\sin x = x$ & $\cos x = 1$ approximately

$\therefore \frac{1}{2} + \frac{\sqrt{3}}{2}x = 0.5033$

$\Rightarrow \frac{\sqrt{3}}{2}x = 0.0033$

$\Rightarrow x = 0.0033 \times \frac{2}{\sqrt{3}}$

$= \frac{66}{17320}$ radians

$= 13'\ 6''$ nearly.

$\therefore \theta = 30^\circ + x = 30^\circ + 13'\ 6''$

$= 30^\circ\ 13'\ 6''$.

(11) Show that the error involved in replacing $\frac{1}{6}(8\sin\theta - \sin 2\theta)$ by θ is approximately $\frac{1}{30}\theta^5$ if θ is small.

Solution:

$$\frac{1}{6}(8\sin\theta - \sin 2\theta) = \frac{8}{6}\sin\theta - \frac{1}{6}\sin 2\theta.$$

$$= \frac{4}{3}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) - \frac{1}{6}\left(2\theta - \frac{8\theta^3}{3!} + \frac{32\theta^5}{5!} + \cdots\right)$$

$$= \frac{4}{3}\theta - \frac{2\theta}{6} - \frac{\theta^3}{3!}\left(\frac{4}{3} - \frac{8}{6}\right) + \frac{\theta^5}{5!}\left(\frac{4}{3} - \frac{32}{6}\right) - \cdots$$

$$= \theta + \frac{\theta^5}{5!}\left(\frac{8-32}{6}\right).$$

$$= \theta + \frac{\theta^5 \cdot \cancel{24}}{1\times 2\times 3\times \cancel{4}\times 5\times \cancel{6}}$$

$$= \theta - \frac{1}{30}\theta^5.$$

Hence the error in taking $\frac{1}{6}(8\sin\theta - \sin 2\theta)$ as θ is $\frac{1}{30}\theta^5$.